

Extremal Graphs for Clique-Paths

Roman Glebov

FU Berlin, Institut für Mathematik, Arnimallee 3, D-14195 Berlin, Germany
glebov@math.fu-berlin.de

December 1, 2011

Abstract

In this paper we deal with a Turán-type problem: given a positive integer n and a forbidden graph H , how many edges can there be in a graph on n vertices without a subgraph H ? How does a graph look like if it has this extremal edge number?

The forbidden graph in this article is a clique-path: a path of length k where each edge is extended to an r -clique, $r \geq 3$. We determine both the extremal number and the extremal graphs for sufficiently large n .

1 Introduction

For integers $n \geq r \geq 1$, we let $T_{n,r}$ denote the *Turán graph*, i.e., the complete r -partite graph on n vertices where each partite set has either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ vertices and the edge set consists of all pairs of vertices from distinct parts. The number of edges in $T_{n,r}$ is denoted by $t_{n,r}$. A K_r represents the complete graph on r vertices.

For a graph G and a vertex $x \in V(G)$, the *neighborhood* of x in G is denoted by $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, or if the underlying graph G is clear from the context, simply $N(x)$. The neighborhood of a subset V' of vertices is the intersection of the neighborhoods of the vertices of V' , $N(V') = \bigcap_{x \in V'} N(x)$. The vertices from $N(x)$ are adjacent to x , we also say that x *sees* these vertices. The *degree* of x in G , denoted by $d_G(x)$ or $d(x)$, is the size of $N_G(x)$. We use $\delta(G)$ to denote the minimum degree in G and $\bar{d}(G)$ for the average degree. A vertex x with degree $d(x) = |V| - 1$ is called a *universal vertex*. If the underlying graph G is clear from the context, we also write $d_A(x)$ for $|N(x) \cap A|$ with $A \subseteq V(G)$. For a subset $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G induced by X . If $X = V(G) \setminus \{v\}$ for some $v \in V(G)$, we also write $G - v$ for $G[X]$. A *matching* in G is a set of edges from $E(G)$, no two of which share a common vertex.

Suppose that we are given a fixed forbidden graph H . A graph is called *H -free*, if it does not contain a copy of H as a subgraph. We are interested in the maximum (*extremal*) number, $ex(n, H)$, of edges an H -free graph on n vertices can have. An H -free graph on n vertices with $ex(n, H)$ edges is called an *extremal graph* for H , or just *H -extremal*.

Mantel [5] determined the extremal number for a triangle, and Turán [7] generalized the result and showed that $T_{n,r-1}$ is the unique extremal graph for the r -clique. Although for bipartite graphs even the asymptotics of the extremal

numbers often remains open, Erdős and Stone [4] proved the asymptotical result $ex(n, H) = (1 + o(1))t_{n, \chi(H)-1}$ for non-bipartite graphs H . The goal in this case is now to determine the precise extremal number and all extremal graphs. Simonovits [6] developed a method to find exact extremal numbers using the stability properties of extremal graphs. A well-known result in this field are for example the octahedron-free graphs determined by Erdős and Simonovits [3]. We denote by $F_{k,r}$ the graph on $(r-1)k+1$ vertices consisting of k r -cliques, which intersect in exactly one common vertex. Erdős, Füredi, Gould and Gunderson [2] determined the extremal number for the $F_{k,3}$ for sufficual large n . Chen, Gould, Pfender and Wei [1] proved the following generalization of the main theorem of [2]:

Theorem 1. *For every $k \geq 1$ and $r > 2$, and for every $n \geq 16k^3r^8$, if a graph G on n vertices has more than*

$$ex(n, K_r) + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even} \end{cases}$$

edges, then G contains a copy of an $F_{k,r}$. Further, the number of edges is best possible.

In this article, we look at k r -cliques intersecting in a different way: let P_k be a k -path with $V(P_k) = \{p_1, \dots, p_{k+1}\}$ and $E(P_k) = \{p_i p_{i+1} : 1 \leq i \leq k\}$. We extend the edges to r -cliques and get a *clique-path* $P_{k,r}$. Formally,

$$V(P_{k,r}) := V(P_k) \cup \{c_{i,j} : 1 \leq i \leq k, 1 \leq j \leq r-2\}$$

and

$$\begin{aligned} E(P_{k,r}) := & E(P_k) \cup \{p_i c_{i,j} : 1 \leq i \leq k, 1 \leq j \leq r-2\} \\ & \cup \{p_{i+1} c_{i,j} : 1 \leq i \leq k, 1 \leq j \leq r-2\} \\ & \cup \{c_{i,a} c_{i,b} : 1 \leq i \leq k, 1 \leq a \leq b \leq r-2\} \end{aligned}$$

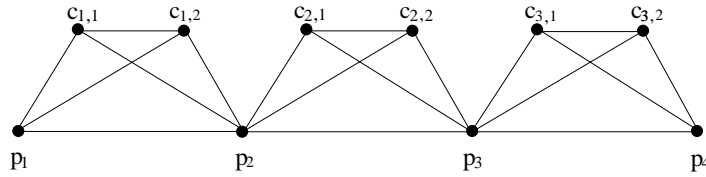


Figure 1: A clique-path $P_{3,4}$

The aim of this paper is to determine the $P_{k,r}$ -extremal graphs. The graph we want to show being extremal $P_{k,r}$ -free on n vertices, n, k, r positive integers with $r \geq 3$ and n sufficiently large, is called $G_{n,k,r}$. It is constructed from the $(r-1)$ -partite Turán graph on $n - \lfloor \frac{k-1}{2} \rfloor$ vertices by adding $\lfloor \frac{k-1}{2} \rfloor$ universal vertices and when k is even, also adding an edge. Formally, let $f := \lfloor \frac{k-1}{2} \rfloor$ and let $G_{n,k,r} := K_f \vee T_{n-f, r-1}$ be the join of an f -clique and the $(r-1)$ -partite Turán graph on $n-f$ vertices with an additional edge, if k is even.

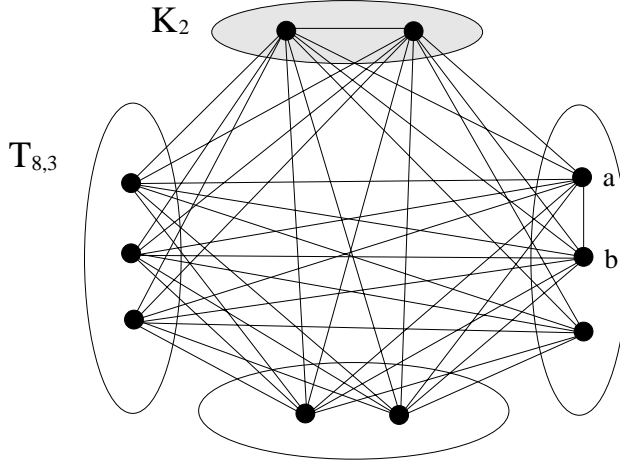


Figure 2: A $G_{10,6,4}$

Notice that if $r - 1$ is not a factor of $n - f$ and k is even, there are two nonisomorphic graphs, both called $G_{n,k,r}$, depending on the size of the set containing ab . Nevertheless, the graphs are quite “similar”, and the small difference does not matter in this article, therefore we will not pay much attention to this fact. The following is the main theorem of this paper.

Theorem 2. *Suppose that G is a $P_{k,r}$ -free graph on n vertices with $n > 16k^8r^{11}$. Then $|E| \leq g_{n,k,r}$ holds and equality occurs if and only if G is isomorphic to a $G_{n,k,r}$.*

We can easily see that $G_{n,k,r}$ does not contain a $P_{k,r}$. Since the Turán graph $T_{n-f,r-1}$ is $r - 1$ -partite, each r -clique in $G_{n,k,r}$ has one vertex from the added K_f or the edge ab . In $P_{k,r}$ there are k r -cliques, and each vertex is contained in two of them at most; the edge ab can be contained only in one r -clique. Thus the length of the longest clique-path with maximal cliques of order r is

$$k - 1 = \begin{cases} 2 \lfloor \frac{k-1}{2} \rfloor & \text{if } k \text{ is odd,} \\ 2 \lfloor \frac{k-1}{2} \rfloor + 1 & \text{for even } k. \end{cases}$$

The remaining part of the proof of Theorem 2 is to show that each $P_{k,r}$ -free graph on n vertices with at least $|E(G_{n,k,r})|$ edges is isomorphic to a $G_{n,k,r}$.

The number of edges in $G_{n,k,r}$ is denoted by $g_{n,k,r} := |E(G_{n,k,r})|$.

For the sake of better readability, we omit the graph G in the notation for the vertex and edge sets and simply write V and E if the underlying graph G is clear from the context. For simplicity of notation, we will identify isomorphic graphs.

2 Reduction to high minimum degree

The next lemma states the theorem in the case of graphs with high minimum degree.

Lemma 1. *Suppose that G is a $P_{k,r}$ -free graph on n vertices with $n > 4k^4r^6$, and with minimum degree $\delta > \frac{r-2}{r-1}n - 1$. Then $|E| \leq g_{n,k,r}$ holds and equality occurs if and only if G is isomorphic to a $G_{n,k,r}$.*

We give the proof of this lemma in Section 3. Using the lemma, we prove Theorem 2.

Proof of Theorem 2. This proof is a so-called “standard backward induction”. Let G be a $P_{k,r}$ -free graph on n vertices with $n > 16k^8r^{11}$ and $|E| \geq g_{n,k,r}$. Suppose there is a vertex $x \in V(G)$ with $d_G(x) < \left\lfloor \frac{r-2}{r-1}n \right\rfloor \leq \delta(G_{n,k,r})$. We initialize $G^n = G$ and define a process by iteratively deleting vertices with minimum degree. We continue the process while $\delta(G^i) < \delta(G_{i,k,r})$, and so during the iterations $|E(G^{i-1})| \geq g_{i-1,k,r} + n - i$. After $n - l$ steps we get a subgraph G^l with $\delta(G^l) \geq \delta(G_{l,k,r}) \geq \left\lfloor \frac{r-2}{r-1}l \right\rfloor$. Note that

$$\binom{l}{2} \geq |E(G^l)| \geq g_{l,k,r} + n - l > n - l + \frac{r-2}{r-1} \frac{l^2}{2} - \frac{r^2}{2}.$$

Hence $l > \sqrt{rn} \geq 4k^4r^6$ and since $|E(G^l)| > g_{l,k,r}$, by Lemma 1 G^l contains a $P_{k,r}$. This is a contradiction to the fact that G^l is $P_{k,r}$ -free as a subgraph of a $P_{k,r}$ -free graph G .

Thus for all vertices $x \in V(G)$ we obtain $d_G(x) > \frac{r-2}{r-1}n - 1$, hence by Lemma 1 $|E| \leq g_{n,k,r}$ holds and equality occurs if and only if G is isomorphic to a $G_{n,k,r}$.

3 The Extremal Graphs for the Clique-Path $P_{k,r}$

In this section, we prove the remaining claim.

Proof of Lemma 1. Suppose that G is a $P_{k,r}$ -free graph on n vertices with $n > 4k^4r^6$, minimum degree $\delta > \frac{r-2}{r-1}n - 1$ and edge number $|E| \geq g_{n,k,r}$. We prove that G is isomorphic to a $G_{n,k,r}$.

We can assume without loss of generality that G has the most edges under all graphs that satisfy these properties. We prove the lemma in a sequence of claims. In the first claim, we see by induction that the whole graph is close to the Turán graph $T_{n,r-1}$. In fact it consists of $r - 2$ independent sets of size roughly $\frac{n}{r-1}$. The union of these sets is called L . In the second claim we show that excluding only a few vertices from the remaining vertices $V \setminus L$, we make the edges in that set independent. We call this set R and the excluded vertices *felons*, since they destroy the structure of L and R . In the third claim and the following proposition we show some technical statements to prove in Claim 4 that there are at most $\left\lfloor \frac{k-1}{2} \right\rfloor$ of these excluded felons. The fifth claim says that there is at most one extra edge inside L or R , if k is even, and none for odd k . Then we maximize the number of edges in G , and we are done.

Claim 1. *V contains a set L that consists of $r - 2$ disjoint independent sets, each with $\left\lceil \frac{1}{r-1}n \right\rceil - kr$ vertices.*

Proof. In [7] and [1], the cases $k = 1$ and $k = 2$ are already proven. We use the result for $k = 2$ to start our induction for $k \geq 3$.

Since the function $g_{n,k,r}$ strictly increases with k (either we get exactly one more edge or one vertex becomes a felon and gets the full degree $n - 1$), by

induction, there is a copy P of a $P_{k-1,r}$ as a subgraph in G . Let x be the vertex corresponding to p_k in P , and let $L = N(x) \setminus V(P)$ be its neighborhood outside P . Obviously, $G[L]$ must be K_{r-1} -free since a K_{r-1} in $G[L]$ would extend P into a $P_{k,r}$ via x (see Figure 3).

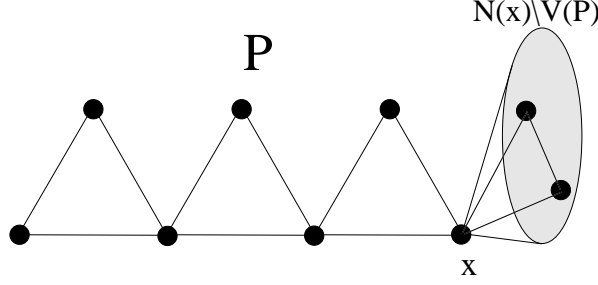


Figure 3: The formation of a $P_{4,3}$ from a $P \simeq P_{3,3}$, if L contains a K_2 .

Let $l = |L|$ be the number of vertices in L , then

$$\begin{aligned}
 l &\geq |N(x)| - |V(P)| \\
 &> \frac{r-2}{r-1}n - 1 - (k-1)(r-1) - 1 \\
 &> \frac{r-2}{r-1}n - kr + r
 \end{aligned} \tag{1}$$

holds.

Notice that for $r = 3$, L is independent, and we are done. We now assume $r \geq 4$. We estimate the average degree within L :

$$\begin{aligned}
 \overline{d}(G[L]) &\geq \delta(G[L]) \\
 &\geq \delta(G) - (n-l) \\
 &> (n - \frac{1}{r-1}n - 1) - (n-l) \\
 &\stackrel{(1)}{>} l - \frac{1}{r-3}l \\
 &\geq \overline{d}(T_{l,r-3}).
 \end{aligned}$$

Thus $|E(G[L])| > ex(l, K_{r-2})$, hence there is an $(r-2)$ -clique K in $G[L]$, $V(K) = \{v_1, \dots, v_{r-2}\}$. We call the $G[L]$ -neighborhoods of $(r-3)$ -subsets of K $L_i = N_{G[L]}(K \setminus \{v_i\})$. Because $G[L]$ is K_{r-1} -free, L_i 's are independent and pairwise disjoint. For any vertex $v \in V(K)$, there are less than $\frac{1}{r-1}n + 1$ vertices in $L \setminus N(v)$, thus each L_i has more than

$$l - (r-3) \left(\frac{1}{r-1}n + 1 \right) > \frac{1}{r-1}n - kr$$

vertices. Together, these neighborhoods form a graph on $r-2$ disjoint independent sets, each with at least $\frac{1}{r-1}n - kr$ vertices. To establish the claim, we remove vertices from L_i 's to have exactly $\left\lceil \frac{1}{r-1}n \right\rceil - kr$ vertices in each and redefine $L := \bigcup L_i$. \square

Note that

$$l = (r-2) \left\lceil \frac{1}{r-1}n \right\rceil - kr(r-2). \quad (2)$$

Claim 2. $V \setminus L$ can be divided into two parts R and F , $V = R \cup F$, so that the edges in $G[R]$ form a matching and $|F| \leq k^2r(r-2) + 2k < k^2r^2$.

Proposition 1. $G[L]$ contains k pairwise disjoint $(r-2)$ -cliques.

Proof. For $k = 3$ note that $l \geq k$, so there are k copies of K_1 in $G[L]$. Now let $k \geq 4$.

Suppose to the contrary that the largest number of pairwise disjoint $(r-2)$ -cliques in L is $k' < k$. Remove a maximal collection of pairwise disjoint $(r-2)$ -cliques and call the resulting set L' . Then

$$\begin{aligned} \delta(G[L']) &> \left(n - \frac{1}{r-1}n - 1 \right) - (n-l) - kr \\ &> l - \frac{1}{r-3}l \\ &\geq \bar{d}(T_{l,r-3}), \end{aligned}$$

thus L' contains a K_{r-2} , leading to a contradiction. \square

Proof of Claim 2. Note that for each $x \in L_i$, there are at most kr vertices outside L_i that are not adjacent to x . Indeed x has at most $\frac{1}{r-1}n + 1 < |L_i| + kr$ non-neighbors.

Let us choose a family of k pairwise disjoint $(r-2)$ -cliques C_1, \dots, C_k . Let F be the set of all vertices in $V \setminus L$, which are not seen by $\bigcup V(C_i)$, $F = (V \setminus L) \setminus N(\bigcup V(C_i))$. We call these vertices *felons*, because they destroy the Turán-like structure of G . Since for each $v \in L$, $|(V \setminus L) \setminus N(v)| \leq kr$, we obtain $f := |F| \leq k^2r(r-2)$. Let R be the set of remaining vertices, $R = V(G) \setminus (L \cup F)$. Now we remove the vertices of disjoint paths of length at least 2 from R greedily until only a matching is left, and add them to F .

If the sum of the lengths of all those removed paths was at least k , we could find a $P_{k,r}$ in G similarly to the Figure 4.

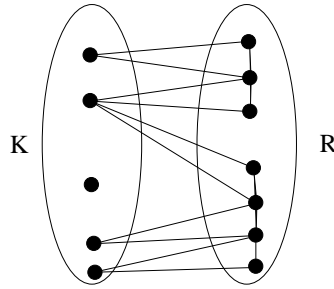


Figure 4: The formation of a $P_{5,3}$ from a P_2 and a P_3 in R , where the sum of the lengths of disjoint paths of length at least 2 in R is at least 5.

Thus we removed at most $2k$ vertices. Notice that the edges in R are pairwise disjoint and there are at most $k^2r(r-2) + 2k < k^2r^2$ felons. \square

Claim 3. V can be divided into two parts S and F , where S consists of $r - 1$ sets, each with at least $\frac{1}{r-1}n - k^2r^2$ vertices, where the edges induced by each of them form a matching, each vertex of F has at least two neighbors in each of these sets, and $e(S, F) \geq \frac{r-2}{r-1}nf + \lfloor \frac{k-1}{2} \rfloor \frac{n}{r-1} - \frac{1}{2} \frac{n}{r-1} - k^4r^4$.

Proof. Recall that $L = L_1 \cup \dots \cup L_{r-2}$. Let $S = R \cup L$ be our set of good vertices. We define $L_{r-1} = R$.

Observation 1. Notice each of the L_i 's has at least $\frac{1}{r-1}n - k^2r^2 + 6$ vertices.

Proof. For $i \leq r - 2$, the observation follows from Claim 1. We can see that

$$\begin{aligned} |R| &\geq n - |L| - k^2r(r-2) - 2k \\ &> \frac{1}{r-1}n - k^2r^2 + 6. \end{aligned}$$

□

For further calculations, we denote by e the number of edges inside all $G[L_i]$'s, and fix the current numbers $f' = f$ and $e' = e$, since f and e change soon. Let us now try to reintegrate some of the felons, that is to insert them into S . To accomplish this, we allow an additional small matching inside the L_i 's and exchange some felons with good vertices. If there exist a felon $x \in F$ and an i with $d_{L_i}(x) \leq 1$, add x to L_i and move the resulting degree-2-vertex from L_i to F , should there be one. Repeat this process, until every felon has at least two neighbors in each L_i .

The process terminates in at most $t := e' + 2f'$ steps, since the value $e + 2f$ is reduced by at least 1 in each iteration.

There are two cases possible for each iteration step: either one felon and at most one edge are added to an L_i , or we exchange a felon with a good vertex and decrease e . Only in a step where f decreases can e increase by one. So at most $k^2r(r-2) + 2k$ edges are added to the L_i 's. Using Observation 1, we have the following observation.

Observation 2. For $i \leq r - 2$, still $|L_i| \geq \frac{1}{r-1}n - kr^2$, and $|L_{r-1}| > \frac{1}{r-1}n - k^2r^2 + 6$. Furthermore, inside each L_i the edges are pairwise disjoint.

There are at most $\frac{r-2}{r-1} \frac{(n-f)^2}{2}$ edges between the L_i 's, $\frac{1}{2} \frac{1}{r-1}n + \frac{1}{2}kr^2$ "old" edges inside R by (2) and Observation 2, $k^2r(r-2) + 2k$ edges that came in during the reintegration of felons and $\binom{f}{2}$ edges inside F . On the other hand, there are more than $\frac{r-2}{r-1} \frac{n^2}{2} - \frac{r^2}{2} + \lfloor \frac{k-1}{2} \rfloor \frac{n}{r-1} - \frac{k^2}{2}$ edges in $G_{n,k,r}$, thus there are more than

$$\begin{aligned} &\frac{r-2}{r-1} \frac{n^2}{2} + \left\lfloor \frac{k-1}{2} \right\rfloor \frac{n}{r-1} - \frac{k^2}{2} - \frac{r^2}{2} \\ &\quad - \frac{r-2}{r-1} \frac{(n-f)^2}{2} - \frac{1}{2} \frac{1}{r-1}n - \frac{1}{2}kr^2 - k^2r(r-2) - 2k \\ &\geq \frac{r-2}{r-1}nf + \left\lfloor \frac{k-1}{2} \right\rfloor \frac{n}{r-1} - \frac{1}{2} \frac{n}{r-1} - k^4r^4 \end{aligned} \tag{3}$$

edges between S and F in G . □

Observation 3. For each $x \in L_i$, there are less than k^2r^2 vertices outside of L_i that are not adjacent to x .

The following two claims will be proven by contradiction using a common principle, so we state a technical proposition.

For any $P \subset V$ and a clique $K \subset (V \setminus P) \cup F$ with exactly one felon and at most one vertex from each L_i , an r -clique C is a P -avoiding extension of K if $K \subseteq C$ and C contains (exactly) one vertex from each $L_i \setminus P$. We use the following statement:

Proposition 2. *For any $P \subset V$ of size $|P| \leq kr$, a P -avoiding extension of K exists provided the felon in K has a degree at least $k^2 r^3$ in each L_i with $L_i \cap K = \emptyset$.*

Proof. Let C be a largest clique in $G[(V \setminus P) \cup F]$ containing K with one vertex from each $L_i \setminus P$. Let $f^{(K)} = K \cap F$ be the felon in K . If there exists an i with $L_i \cap C = \emptyset$, then since C is maximal, by Observation 3

$$\begin{aligned} |N_S(f^{(K)})| &\leq \left| \left(L_i \setminus N \left(C \setminus \{f^{(K)}\} \right) \right) \cup P \right| \\ &< k^2 r^3, \end{aligned}$$

proving the proposition. \square

Claim 4. *There are at most $\lfloor \frac{k-1}{2} \rfloor$ felons, $f \leq \lfloor \frac{k-1}{2} \rfloor$.*

Assume for the sake of contradiction that there are more than $\lfloor \frac{k-1}{2} \rfloor$ felons in G , $f \geq \lfloor \frac{k+1}{2} \rfloor$. We take the $\lfloor \frac{k+1}{2} \rfloor$ felons $f_1, \dots, f_{\lfloor \frac{k+1}{2} \rfloor}$ with the highest S -degrees $d_S(f_1) \leq \dots \leq d_S(f_{\lfloor \frac{k+1}{2} \rfloor})$. We call $C \simeq P_{2,r}$ a *connector* between felons x and y , if $x, y \in V(C)$ and $xy \notin E(C)$. (See Figure 5.)

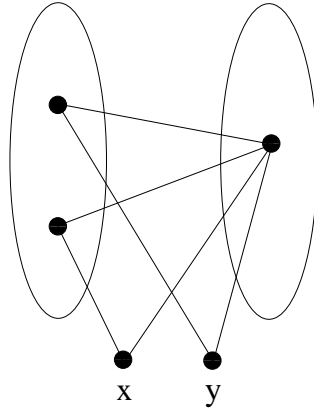


Figure 5: The formation of a connector between two felons x and y for $r = 3$.

Here is our plan: we attach an r -clique $D_0^{(2)}$ to f_1 , then find a connector with cliques $D_i^{(1)}, D_i^{(2)}$ between f_i and f_{i+1} for $1 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$. In the end, we attach an r -clique $D_{\lfloor \frac{k+1}{2} \rfloor}^{(1)}$ to $f_{\lfloor \frac{k+1}{2} \rfloor}$ and get the forbidden clique-path $P_{k,r}$, contradicting the $P_{k,r}$ -freeness of G . (See Figure 6.) This contradiction shall show that $|F| \leq \lfloor \frac{k-1}{2} \rfloor$.

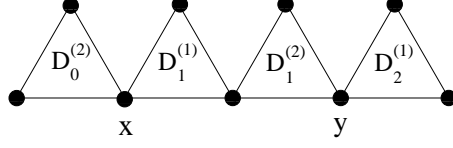


Figure 6: The formation of a $P_{4,3}$ by two felons x and y , their connector and attached triangles.

Of course we have to pay attention so the attached connectors and K_{r-1} are only intersecting where they are supposed to.

To be able to use Proposition 2, we lower bound the degrees of f_i . By (3) we know that

$$e\left(F \setminus \left\{f_3, \dots, f_{\lfloor \frac{k+1}{2} \rfloor}\right\}, S\right) > \frac{r-2}{r-1}n \left(f - \left\lfloor \frac{k-3}{2} \right\rfloor\right) + \frac{1}{2} \frac{n}{r-1} - k^4 r^4,$$

so since $d_S(f_2)$ is maximal within $F \setminus \left\{f_3, \dots, f_{\lfloor \frac{k+1}{2} \rfloor}\right\}$, we have

$$\begin{aligned} d_S(f_2) &> \frac{\frac{r-2}{r-1}nf + \left\lfloor \frac{k-1}{2} \right\rfloor \frac{n}{r-1} - \frac{1}{2} \frac{n}{r-1} - k^4 r^4 - n \left\lfloor \frac{k-3}{2} \right\rfloor}{f - \left\lfloor \frac{k-3}{2} \right\rfloor} \\ &= \frac{\frac{r-2}{r-1}n \left(f - \left\lfloor \frac{k-3}{2} \right\rfloor\right) + \frac{1}{2} \frac{n}{r-1} - k^4 r^4}{f - \left\lfloor \frac{k-3}{2} \right\rfloor} \\ &> \frac{r-2}{r-1}n + \frac{1}{2} \frac{n}{k^2 r^3} - k^2 r^2. \end{aligned} \quad (4)$$

To complete this estimation, the number of neighbors of f_2 in any L_i by Observation 3 is

$$\begin{aligned} d_{L_i}(f_2) &\geq |L_i| - (|S| - d_S(f_2)) \\ &\geq \frac{1}{r-1}n - k^2 r^2 - \left(n - \frac{r-2}{r-1}n - \frac{1}{2} \frac{n}{k^2 r^3} + k^2 r^2\right) \\ &= \frac{1}{2} \frac{n}{k^2 r^3} - 2k^2 r^2 \\ &\geq \frac{1}{4} \frac{n}{k^2 r^3} - 4k^2 r^3 + k^2 r^3 \\ &> k^2 r^3. \end{aligned} \quad (5)$$

Due to the definition of f_2 , $d_{L_i}(f_j) > k^2 r^3$ for $j \geq 2$.

Since we only need the connectors and the attached cliques to find the forbidden $P_{k,r}$, we add at most $|V(P_{k,r})| < kr$ vertices to the avoided set P . Thus, by (4) and Proposition 2 we can make the following corollary.

Corollary 1. *A P -avoiding extension always exists for felons f_i with $i \geq 2$.*

Let us choose one neighbor x of f_1 in the set L_j with the fewest neighbors of f_1 . Because of $\delta_G > \frac{r-2}{r-1}n - 1$, there are less than $\frac{1}{r-1}n + 1$ vertices outside of the neighborhood of f_1 , and at most half of them (that is $\frac{1}{2} \frac{1}{r-1}n + 1$) may be found in each of the other L_i 's with the exception of L_j .

Since each of these L_i 's has at least $\frac{1}{r-1}n - k^2r^2 + 6$ vertices, at least

$$\begin{aligned} \frac{1}{r-1}n - k^2r^2 + 6 - \frac{1}{2} \frac{1}{r-1}n - 1 &> \frac{1}{2} \frac{1}{r-1}n - k^2r^2 \\ &\geq k^2r^3 \end{aligned}$$

of them are in the neighborhood of f_1 , hence we obtain the following statement.

Observation 4. *We can find more than k^2r^3 neighbors of f_1 in any L_i with $i \neq j$.*

Hence by Proposition 2 we can take a \emptyset -avoiding extension $D_0^{(2)}$ of $\{f_1\}$ and call its vertex set $P = V(D_0^{(2)})$.

To construct $D_1^{(1)}$ and $D_1^{(2)}$, we distinguish two cases:

Case 1. There is a common neighbor v of f_1 and f_2 in $L_j \setminus P$. Then we find a connector between f_1 and f_2 the following way. We take a P -avoiding extension $D_1^{(1)}$ of $\{f_1, v\}$ and redefine $P := P \cup \left(V(D_1^{(1)}) \setminus \{v\}\right)$. Then we take a P -avoiding extension $D_1^{(2)}$ of $\{f_2, v\}$ and add it to $P := P \cup V(D_1^{(2)})$. Using Observation 4 and Proposition 2, we find the $D_1^{(1)}$. The existence of $D_1^{(2)}$ is asserted by Corollary 1.

Obviously, D_1 and D_2 only intersect in v , so they form a connector between f_1 and f_2 we searched for.

Case 2. All the (more than k^2r^2) common neighbors of f_1 and f_2 in S are outside of $L_j \setminus P$. We proceed similar to the first case with one difference: we take a neighbor y of f_1 in $L_j \setminus P$, and then find a common neighbor v of f_1 and f_2 so that $yv \in E$. To prove that such a vertex exists we make the following proposition.

Proposition 3. *For any $P \subset V$ of size $|P| < kr$ and $i, j \leq \lfloor \frac{k+1}{2} \rfloor$, there are more than k^2r^2 common neighbors of f_i and f_j in $S \setminus P$.*

Proof. Using (4) we have

$$\begin{aligned} d_{S \setminus P}(f_i) + d_{S \setminus P}(f_j) &\geq d_{S \setminus P}(f_1) + d_{S \setminus P}(f_2) \\ &> d_S(f_1) - kr + d_S(f_2) - kr \\ &\geq \delta - f + d_S(f_2) - 2kr \\ &> \frac{r-2}{r-1}n - 1 + \frac{r-2}{r-1}n + \frac{1}{2} \frac{n}{k^2r^3} - 2k^2r^2 - 2kr \\ &\geq \frac{1}{2}n + \frac{1}{2}n + \frac{1}{2} \frac{n}{k^2r^3} - 4k^2r^2 + k^2r^2 \\ &\geq n + k^2r^2 \\ &> |S| + k^2r^2. \end{aligned}$$

□

Since by Observation 3, there are at most k^2r^2 vertices outside of L_j that are not seen by y , we find v . Then we choose a P -avoiding extension $D_1^{(1)}$ of the triangle $\{f_1, y, v\}$, redefine $P := P \cup \left(V(D_1^{(1)}) \setminus \{v\}\right)$, then take a P -avoiding extension $D_1^{(2)}$ of $\{f_2, v\}$, add it to $P := P \cup V(D_1^{(2)})$ and we are done.

It is easier to find connectors between f_i and f_{i+1} , $2 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$. By Proposition 3, we can find a common neighbor v of f_i and f_{i+1} in S , even after having added all the previous connectors and the attached r -clique to P earlier. Then we take a P -avoiding extension $D_i^{(1)}$ of $\{f_i, v\}$, redefine $P := P \cup V(D_i^{(1)} - v)$, and a P -avoiding extension $D_i^{(2)}$ of $\{f_{i+1}, v\}$, $P := P \cup V(D_i^{(2)})$ and get a connector between f_i and f_{i+1} following the same argument as in Case 1.

Finally, for an even k we can attach a P -avoiding extension $D_{\lfloor \frac{k+1}{2} \rfloor}^{(1)}$ of $\{f_{\lfloor \frac{k+1}{2} \rfloor}\}$ to $f_{\lfloor \frac{k+1}{2} \rfloor}$ by Corollary 1.

Hereby we get the forbidden $P_{k,r}$, hence the supposition that there are at least $\lfloor \frac{k+1}{2} \rfloor$ felons was wrong; consequently there are at most $\lfloor \frac{k-1}{2} \rfloor$ of them. \square

Claim 5. *There is at most one edge inside the L_i 's, if k is even, and the L_i 's are independent for odd k .*

Proof. By Claim 3, there are at least $\frac{r-2}{r-1}nf + \lfloor \frac{k-1}{2} \rfloor \frac{n}{r-1} - \frac{1}{2} \frac{n}{r-1} - k^4 r^4$ edges between F and S , thus there are at least $\lfloor \frac{k-1}{2} \rfloor$ felons. Hence, by Claim 3 we have exactly $\lfloor \frac{k-1}{2} \rfloor$ felons $f_1, \dots, f_{\lfloor \frac{k-1}{2} \rfloor}$. Let us assume $d_S(f_1) \leq \dots \leq d_S(f_{\lfloor \frac{k-1}{2} \rfloor})$. Now f_1 has at least an S -degree $d_S(f_1) > \frac{r-2}{r-1}n + \frac{1}{2} \frac{n}{r-1} - k^4 r^4$ (and so do all the other f_i 's). Thus, any two felons f_i and f_j have more than $kr + 4$ common neighbors. Hence, we may delete any four vertices from S , initialize $P = \emptyset$ and do a construction similar to the one in Claim 4. For i from 1 to $\lfloor \frac{k-3}{2} \rfloor$, we take a common neighbor v of f_i and f_{i+1} in $S \setminus P$, find a P -avoiding extension $D_i^{(1)}$ of $\{f_i, v\}$, redefine $P := P \cup V(D_i^{(1)} - v)$, and then take a P -avoiding extension $D_i^{(2)}$ of $\{f_{i+1}, v\}$ and add it to $P := P \cup V(D_i^{(2)})$. Again the existence of the extensions is asserted by Proposition 2. This way, we constructed a P with $P_2[\lfloor \frac{k-1}{2} \rfloor - 2, r] \subseteq G[P]$ only with connectors from each f_i to the successor f_{i+1} .

Our aim is to show that there are not too many edges inside the L_i 's. We only deal with odd k now, since the case of an even k is similar. So what happens if we would find an edge ab inside an L_i ? We can deport a to the felons, $L_i := L_i \setminus \{a\}$, $F := F \cup \{a\}$, and find a common neighbor v of a , b and f_1 in S . Now we attach a connector to our clique-path by finding a P -avoiding extension $D_0^{(1)}$ of $\{f_1, v\}$, redefining $P := P \cup V(D_0^{(1)} - v)$, taking a P -avoiding extension $D_0^{(2)}$ of $\{a, b, v\}$, and modifying $P := P \cup V(D_0^{(2)})$. (See Figure 7.)

Now we can attach a P -avoiding extension $D_{\lfloor \frac{k-1}{2} \rfloor}^{(1)}$ of $\{f_{\lfloor \frac{k-1}{2} \rfloor}\}$ to the clique-path and get a forbidden $P_{k,r}$. But this, as we have already seen, leads us to a contradiction. Thus the L_i s are independent.

Similarly to that we get at most one edge in the L_i s for an even k . \square

Since G has as many edges as possible, there have to be all the edges inside F and between F and S , and S has to be a $T_{n-f, r-1}$ (and, of course, the one edge must be there if k is even), thus G is isomorphic to a $G_{n, k, r}$. \square

To avoid tedious calculations, I did not attempt to lower the bound $n \geq 16k^8 r^{11}$ in the proof, although I strongly believe the bound can be lowered substantially.

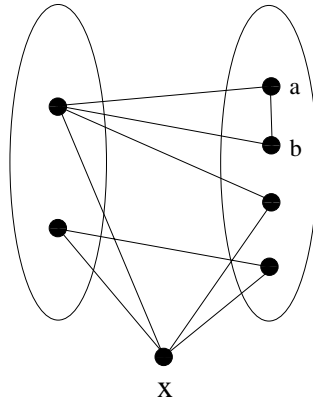


Figure 7: The formation of a $P_{3,3}$ with a (red) felon and a (green) extra edge.

Acknowledgement. My thanks are due to Florian Pfender for suggesting this problem and also for several fruitful comments and discussions. I would also like to thank Konrad Engel, Thomas Kalinowski, and Tibor Szabó, whose comments helped improve this paper considerably.

References

- [1] G. Chen, R. J. Gould, F. Pfender, B. Wei, Extremal Graphs for Intersecting Cliques, *J. Combin. Theory Ser. B* **89**, No. 2 (2003), 159–171.
- [2] P. Erdős, Z. Füredi, R. J. Gould, D. S. Gunderson, Extremal Graphs for Intersecting Triangles, *J. Combin. Theory Ser. B* **64**, No. 1 (1995), 89–100.
- [3] P. Erdős, M. Simonovits, An extremal graph problem, *Acta Math. Acad. Sci. Hung.* **22** (3-4) (1971), 275–282.
- [4] P. Erdős, A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math.* **52** (1946), 1089–1091.
- [5] W. Mantel, Problem 28, *Wiskundige Opgaven* **10** (1906), 60–61.
- [6] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, *Theory of Graphs, Proc. Colloq., Tihany (1966)*, 279–319, Academic Press, New York
- [7] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* **48** (1941), 436–452. [in Hungarian] (Also see, *On the theory of graphs, Colloq. Math.* **3** (1954), 19–30.)